An analytical application of Niedermayer's algorithm to the Edwards-Anderson model: analytical results for the multicritical point on the Nishimori line

Chiaki Yamaguchi

Kosugichou 1-359, Nakahara-ku, Kawasaki-si, Kanagawa 211-0063, Japan

We apply analytically Niedermayer's algorithm to the Edwards-Anderson model on random graphs with arbitary degree distributions. The results for the multicritical point on the Nishimori line are shown. The results are shown by applying a criterion for spin models on the random graphs with arbitary degree distributions. The application of Niedermayer's algorithm makes the size of the Fortuin-Kasteleyn cluster small and shifts the percolation threshold. The results for the $\pm J$ model and the Gaussian model are respectively shown. In the present article, it is respectively shown for the $\pm J$ model and the Gaussian model that, by adjusting an introduced parameter for Niedermayer's algorithm, the percolation threshold obtained in the present article agrees with the location of the multicritical point. We naively estimate the locations of the multicritical points for the $\pm J$ model and the Gaussian model on the randam graphs with arbitary degree distributions.

KEYWORDS: spin glass, percolation, complex network, the Nishimori line, cluster algorithm

1. Introduction

Study of spin models on complex network has been important[1] with the development of study of complex network. In the present article, random graphs with arbitary degree distributions are adopted as examples of the complex network. In the present article, behavior of spins on no growing network is investigated.

We investigate the Edwards-Anderson model[2] as an Ising spin-glass model. The understandings for the Edwards-Anderson model on random graphs and on the Bethe lattice are still uncompleted.[1,3,4] In the present article, the $\pm J$ model and the Gaussian model in the Edwards-Anderson model are respectively investigated. The Nishimori line in the Edwards-Anderson model is a line in the phase diagram for the exchange interactions and the temperature. The internal energy, the upper bound of the specific heat, and so forth are exactly calculated on the Nishimori line in the Edwards-Anderson model.[5–9] The location of the multicritical point in the Edwards-Anderson model on the square lattice is conjectured, and it is shown that the conjectured value is in good agreement with the other numerical estimates.[10] In the present article, the results on the Nishimori line are shown.

There is a case for occuring a percolation transition of network that the network is devided into a lot of networks by deleting nodes and/or links on a network. We define the percolation transition as the percolation transition of network in the present article. There is a case for occuring a percolation transition of cluster that the cluster consists of bonds put between spins becomes a giant component. The bond is fictitious. We define the percolation transition as the percolation transition of cluster in the present article. In the present article, the percolation transition of cluster on a complex network is mentioned.

Powerful Monte Carlo methods, using the Fortuin-Kasteleyn (FK) cluster in the FK random cluster model,[11] that are called the cluster algorithms have been proposed.[12–16] The Swendsen-Wang (SW) algorithm,[12] using the FK cluster, may be one of the most popular cluster algorithms for classical spin models. In the Edwards-Anderson model that has a conflict in the interactions, the percolation transition point of the FK cluster by the SW algorithm disagrees with the phase transition point.[17–19] The disagreement makes the cluster algorithm inefficient. The application of Niedermayer's algorithm[20] makes the cluster small and shifts the percolation threshold. Originally, Niedermayer's algorithm is proposed as a Monte Carlo method for exact numerical estimates. The efficiency of Niedermayer's algorithm in the Edwards-Anderson model is shown by Liang.[21] In the present article, we apply Niedermayer's algorithm as an exact analytical method.

In addition to the application of a gauge transformation,[5,23] our results are shown by applying a criterion[19] for spin models on the random graphs with arbitary degree distributions. In Ref.,[19] by applying the criterion with the gauge transformation, the percolation thresholds of the FK cluster for the Edwards-Anderson model on the random graphs with arbitary degree distributions are analytically calculated on the Nishimori line.

In the present article, we naively estimate the locations of the multicritical points for the $\pm J$ model and the Gaussian model on the random graphs with arbitary degree distributions.

The present article is organised as follows. First, a complex network model and the Edwards-Anderson model are described. Niedermayer's algorithm is described. A criterion for percolation of cluster is described. The percolation threshold is shown for the $\pm J$ model. The percolation threshold is shown for the Gaussian model. The present article is summarized.

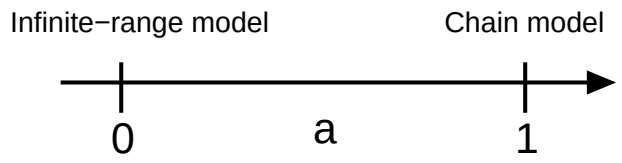

Fig. 1. A relation between the aspect a of the network and the model on the corresponding network of the aspect a.

2. A complex network model and the Edwards-Anderson model

Network consists of nodes and links connected between nodes. In the present article, the network models are random graphs with arbitary degree distributions. The network has no correlation between nodes. The node degree, k, is generated according to a distribution of the node degree, $p(k)$. The links are randomly connected between nodes.

We define a variable $b(i,j)$ where $b(i,j)$ gives one when node i and node j is connected by a link and gives zero when node i and node j is not connected by the link. The degree $k(i)$ of node i is given by

$$k(i) = \sum_j b(i,j)\,. \tag{1}$$

The average of the node degree for links, $< k >_N$, is given by

$$< k >_N = \frac{1}{N}\sum_i^N k(i) \tag{2}$$

where $<>_N$ denotes the average over the whole network. N is the number of nodes. The average of the square of the node degree for links, $< k^2 >_N$, is given by

$$< k^2 >_N = \frac{1}{N}\sum_i^N k^2(i)\,. \tag{3}$$

We define

$$a = \frac{2 < k >_N}{< k^2 >_N} \tag{4}$$

where a represents an aspect of the network.

Fig. 1 shows a relation between the aspect a of the network and the model on the corresponding network of the aspect a. A network is almost the complete graph when a is close to zero, and the model on the network is almost an infinite-range model. We define a coordination number as z. $p(k) = \delta_{k,z}$ where δ is the Dirac delta function. The coordination number z is two when the aspect a of the network is one. The network consists of many cycle graphs when z is two. The model on the network consists of many chain models. In the Erdős-Rényi (ER) random graph model and in the Gilbert model, the distribution of node degree is the Poisson distribution.[1] Therefore, in the ER random graph model and in the Gilbert model, $< k^2 >_N = < k >_N (< k >_N + 1)$. The ER random graph model is a network model that the network consists of the fixed number of nodes and the fixed number of links, and the links are randomly connected between the nodes. The Gilber model is a network model that link between nodes is connected with a given probability.

The Hamiltonian for the Edwards-Anderson model, $\mathcal{H}$, is given by

$$\mathcal{H} = -\frac{1}{2}\sum_i^N \sum_{\{j|b(i,j)=1\}} J_{ij} S_i S_j \tag{5}$$

where $S_i = \pm 1$. J_{ij} is the strength of the exchange interaction between the spin on node i and the spin on node j. The value of J_{ij} is given with the distribution $P(J_{ij})$. The $\pm J$ model and the Gaussian model are respectively given with the difference of $P(J_{ij})$. For the $\pm J$ model, the distribution $P^{(\pm J)}(J_{ij})$ of J_{ij} is given by

$$P^{(\pm J)}(J_{ij}) = p\,\delta_{J_{ij},J} + (1-p)\,\delta_{J_{ij},-J} \tag{6}$$

where $J > 0$. p is the probability that the interaction is the ferromagnetic interaction ($J_{ij} = J$). $1-p$ is the probability that the interaction is the antiferromagnetic interaction ($J_{ij} = -J$). For the Gaussian model, the distribution $P^{(\mathrm{Gaussian})}(J_{ij})$ of J_{ij} is given by

$$P^{(\mathrm{Gaussian})}(J_{ij}) = \frac{1}{\sqrt{2\pi J^2}}\, e^{-\frac{(J_{ij}-J_0)^2}{2J^2}}\,. \tag{7}$$

The average of J_{ij} is $[J_{ij}]_R = J_0$ where $[\,]_R$ denotes the random configuration average. The variance of J_{ij} is $[J_{ij}^2]_R - [J_{ij}]_R^2 = J^2$.

In the present article, for calculating thermodynamic quantities, a gauge transformation:

$$J_{ij} \to J_{ij}\sigma_i\sigma_j\,, \quad S_i \to S_i\sigma_i \tag{8}$$

is used where $\sigma_i = \pm 1$. Using the gauge transformation, $\mathcal{H} \to \mathcal{H}$ and $P(J_{ij}) \to P(J_{ij}\sigma_i\sigma_j)$. This transformation has no effect on thermodynamic quantities.[23]

3. Niedermayer's algorithm

We define a parameter ξ where the range of ξ is

$$0 \le \xi \le 1\,. \tag{9}$$

ξ is adjustable. The bond is put between spins with the corresponding probability $P_{\mathrm{bond}}(S_i, S_j, J_{ij})$ of the spin states and the strength of the exchange interaction. The probability $P_{\mathrm{bond}}(S_i, S_j, J_{ij})$ is

$$P_{\mathrm{bond}}(S_i, S_j, J_{ij}) = 1 - e^{-\beta\xi J_{ij} S_i S_j - \beta\xi|J_{ij}|}\,. \tag{10}$$

where β is the inverse temperature and $\beta = 1/k_B T$. k_B is the Boltzmann constant and T is the temperature. By connecting the bonds, the clusters by Niedermayer's algorithm are generated. We define an index of the cluster on node i as $c(i)$. We define the probability for flipping the cluster as P_{flip}. P_{flip} is

$$P_{\mathrm{flip}} = e^{-2\beta(1-\xi)\sum_{<ij|c(i)\neq c(j)>} J_{ij} S_i S_j} \tag{11}$$

where $< xy >$ denotes the nearest neighbor pairs connected by links, and $< xy\,|\,c(x) \neq c(y) >$ denotes the nearest neighbor pairs connected by links on condition that $c(x) \neq c(y)$. The algorithm is the Metropolis method[24] when $\xi = 0$. The algorithm is the SW algorithm[12] when $\xi = 1$.

The thermodynamic quantity of the bond put between the spin on node i and the spin on node j,

$[< b_{\rm bond}(i,j) >_T]_R$, is

$$[< b_{\rm bond}(i,j) >_T]_R = [< P_{\rm bond}(S_i, S_j, J_{ij}) >_T]_R \quad (12)$$

where $<>_T$ denotes the thermal average. The thermodynamic quantity of the node degree for bonds at node i, $[< k_{\rm bond}(i) >_T]_R$, is

$$[< k_{\rm bond}(i) >_T]_R = [< \sum_{\{j|b(i,j)=1\}} P_{\rm bond}(S_i, S_j, J_{ij}) >_T]_R \,. \quad (13)$$

The thermodynamic quantity of the square of the node degree for bonds at node i, $[< k^2_{\rm bond}(i) >_T]_R$, is

$$[< k^2_{\rm bond}(i) >_T]_R = [< \sum_{\{j|b(i,j)=1\}} \sum_{\{l|b(i,l)=1\}} P_{\rm bond}(S_i, S_j, J_{ij}) \times P_{\rm bond}(S_i, S_l, J_{il})(1-\delta_{j,l}) + \sum_{\{j|b(i,j)=1\}} P_{\rm bond}(S_i, S_j, J_{ij}) >_T]_R \,. \quad (14)$$

The thermodynamic quantity of the node degree for bonds, $[< k_{\rm bond} >_T]_R$, is

$$[< k_{\rm bond} >_T]_R = \frac{1}{N}\sum_i^N [< k_{\rm bond}(i) >_T]_R \,. \quad (15)$$

The thermodynamic quantity of the square of the node degree for bonds, $[< k^2_{\rm bond} >_T]_R$, is

$$[< k^2_{\rm bond} >_T]_R = \frac{1}{N}\sum_i^N [< k^2_{\rm bond}(i) >_T]_R \,. \quad (16)$$

4. A criterion for percolation of cluster

The criterion of the percolation of cluster for spin models on the random graphs with arbitary degree distributions is given by

$$[< k^2_{\rm bond} >_T]_R \geq 2[< k_{\rm bond} >_T]_R \quad (17)$$

.[19] Ineq. (17) is given by the inequality when the cluster is percolated. Ineq. (17) is given by the equality when the cluster is at the percolation transition point. It is anticipated that ineq. (17) is true for sufficiently large number of nodes in the case that the bond does not depend on the size of the degree $k(i)$.

We define a variable for the inverse temperature β as $\rho(\beta)$. We set

$$0 < \rho(\beta) \leq 1 \,. \quad (18)$$

We consider a case that $[< b_{\rm bond}(i,j) >_T]_R$, $[< k_{\rm bond}(i) >_T]_R$, and $[< k^2_{\rm bond}(i) >_T]_R$ are respectively written in

$$[< b_{\rm bond}(i,j) >_T]_R = \rho(\beta) \,, \quad (19)$$

$$[< k_{\rm bond}(i) >_T]_R = \rho(\beta)\, k(i) \,, \quad (20)$$

$$[< k^2_{\rm bond}(i) >_T]_R = \rho^2(\beta)\, k(i)[k(i)-1] + \rho(\beta)\, k(i) \,. \quad (21)$$

In the case, it is implied that the bias for the size of $k(i)$ does not appear in the statistical results of bonds put between spins. Therefore, in the case that $[< b_{\rm bond}(i,j) >_T]_R$, $[< k_{\rm bond}(i) >_T]_R$, and $[< k^2_{\rm bond}(i) >_T]_R$ are respectively written in eqs. (19, 20, 21), the bond does not depend on the size of $k(i)$.

5. The $\pm J$ model

The distribution $P^{(\pm J)}(J_{ij})$ of J_{ij} for the $\pm J$ model is, using eq. (6), given by

$$P^{(\pm J)}(J_{ij}) = \frac{e^{\beta_{\rm P} J_{ij}}}{2\cosh(\beta_P J)} \,, \quad J_{ij} = \pm J \quad (22)$$

where β_P is given by

$$\beta_P = \frac{1}{2J} \ln \frac{p}{1-p} \quad (23)$$

.[5]

The thermodynamic quantity of the bond put between the spin on node i and the spin on node j, $[< b_{\rm bond}(i,j) >_T]_R^{(\pm J)}$, is, using eqs. (8, 10, 12, 22) when $\beta = \beta_P$,

$$\begin{aligned} & [< b_{\rm bond}(i,j) >_T]_R^{(\pm J)} \\ = & \sum_{\{J_{lm}\}} \prod_{<lm>} P^{(\pm J)}(J_{lm}) \times \\ & \frac{\sum_{\{S_l\}} P_{\rm bond}(S_i, S_j, J_{ij})\, e^{\beta_P \sum_{<lm>} J_{lm} S_l S_m}}{\sum_{\{S_l\}} e^{\beta_P \sum_{<lm>} J_{lm} S_l S_m}} \\ = & \frac{1}{2^N [2\cosh(\beta_P J)]^{N_B}} \times \\ & \sum_{\{J_{lm}\}} \sum_{\{S_l\}} P_{\rm bond}(S_i, S_j, J_{ij})\, e^{\beta_P \sum_{<lm>} J_{lm} S_l S_m} \\ = & \frac{1 - e^{-2\beta_P \xi J}}{1 + e^{-2\beta_P J}} \end{aligned} \quad (24)$$

where N_B is the number of all links, and $N_B = N < k >_N /2$. We set

$$\rho^{(\pm J)}(\beta_P) = \frac{1 - e^{-2\beta_P \xi J}}{1 + e^{-2\beta_P J}} \,. \quad (25)$$

The thermodynamic quantity of the node degree for bonds at node i, $[< k_{\rm bond}(i) >_T]_R^{(\pm J)}$, is, using eqs. (8, 10, 13, 22, 25) when $\beta = \beta_P$,

$$[< k_{\rm bond}(i) >_T]_R^{(\pm J)} = \rho^{(\pm J)}(\beta_P)\, k(i) \,. \quad (26)$$

The thermodynamic quantity of the square of the node degree for bonds at node i, $[< k^2_{\rm bond}(i) >_T]_R^{(\pm J)}$, is, using eqs. (8, 10, 14, 22, 25) when $\beta = \beta_P$,

$$[< k^2_{\rm bond}(i) >_T]_R^{(\pm J)} = [\rho^{(\pm J)}(\beta_P)]^2\, k(i)[k(i)-1] + \rho^{(\pm J)}(\beta_P)\, k(i) \,. \quad (27)$$

Eqs. (24, 25, 26, 27) are formulated as eqs. (19, 20, 21), respectively. Therefore, the bond does not depend on the size of $k(i)$. Using eqs. (15, 16, 17, 25, 26, 27), we obtain

$$(2-a)(1 - e^{-2\beta_P \xi J}) \geq a(1 + e^{-2\beta_P J}) \,. \quad (28)$$

Ineq. (28) is given by the inequality when the cluster is percolated. Ineq. (28) is given by the equality when

the cluster is at the percolation transition point. From eq. (28), the percolation threshold is shifted by adjusting ξ while eq. (23) holds. Therefore, it is realized that, by adjusting ξ, the percolation threshold agrees with the location of the multicritical point since, by adjusting ξ, the percolation threshold is shifted along the Nishimori line.

We naively estimate the location of the multicritical point for the $\pm J$ model on the random graphs with arbitary degree distributions. We assume that ξ is the function of a. The network is at the percolation point when a is one.[19] The exchange interaction is only the ferromagnetic interaction when a is one. Therefore, ξ is one when a is one since the algorithm is the SW algorithm[12] when ξ is one. The percolation transition point of the cluster by the SW algorithm agrees with the phase transition point in ferromagnetic spin models.[25] We set $< k >_N = N-1$, $< k^2 >_N = (N-1)^2$, $a = 2/(N-1)$, and $J \to J/\sqrt{N}$. From the settings, the model on the network becomes the infinit-range $\pm J$ model. Using eq. (28), we obtain

$$(N-2)(1-e^{-\frac{2\beta_P \xi J}{\sqrt{N}}}) \geq 1 + e^{-\frac{2\beta_P J}{\sqrt{N}}} . \tag{29}$$

From numerical estimation of eq. (29), $\xi = O(\frac{1}{\sqrt{N}})$ since β_P has a finite value only when $\xi = O(\frac{1}{\sqrt{N}})$. Therefore, we naively estimate

$$\xi^* = \sqrt{a} . \tag{30}$$

Using eqs. (28, 30), we obtain

$$(2-a)(1-e^{-2\sqrt{a}\beta_P J}) \geq a(1+e^{-2\beta_P J}) . \tag{31}$$

Ineq. (31) gives the location of the multicritical point. Using eqs. (23, 29, 30), the location of the multicritical point for the infinit-range $\pm J$ model is numerically estimated as $p^* \sim 0.804 \sim 1/(1+\exp(-\sqrt{2}))$ and $T_P^* \sim 1.414 \sim \sqrt{2}$ when $J/k_B = 1$ is set. Ineq. (31) is obtained based on some assumptions. We have assumed that ξ is written as a simple form of a, and have assumed that the location of the multicritical point for the infinit-range $\pm J$ model has a finite value.

6. The Gaussian model

The distribution $P^{(\mathrm{Gaussian})}(J_{ij})$ of J_{ij} for the Gaussian model is given in eq. (7). We set

$$\beta_P = \frac{J_0}{J^2} \tag{32}$$

.[5]

The thermodynamic quantity of the bond put between the spin on node i and the spin on node j, $[< b_{\mathrm{bond}}(i,j) >_T]_R^{(\mathrm{Gaussian})}$, is, using eqs. (7, 8, 10, 12, 32) when $\beta = \beta_P$,

$$\begin{aligned}
& [< b_{\mathrm{bond}}(i,j) >_T]_R^{(\mathrm{Gaussian})} \\
= & \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{<lm>} dJ_{lm} \prod_{<lm>} P^{(\mathrm{Gaussian})}(J_{lm}) \times \\
& \frac{\sum_{\{S_l\}} P_{\mathrm{bond}}(S_i, S_j, J_{ij})\, e^{\beta_P \sum_{<lm>} J_{lm} S_l S_m}}{\sum_{\{S_l\}} e^{\beta_P \sum_{<lm>} J_{lm} S_l S_m}}
\end{aligned}$$

$$\begin{aligned}
= & \frac{1}{2^N (2\pi J^2)^{N_B/2}} e^{-N_B \frac{J_0^2}{2J^2}} \times \\
& \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{<lm>} dJ_{lm} \sum_{\{S_l\}} P_{\mathrm{bond}}(S_i, S_j, J_{ij}) \times \\
& e^{-\sum_{<lm>} \frac{J_{lm}^2}{2J^2} + \beta_P \sum_{<lm>} J_{lm} S_l S_m} \\
= & \frac{1}{2}[1 + \mathrm{erf}(\frac{\beta_P J}{\sqrt{2}})] - \\
& \frac{1}{2}\{1 + \mathrm{erf}[\frac{\beta_P J}{\sqrt{2}}(1-2\xi)]\}\, e^{-2(\beta_P J)^2 \xi(1-\xi)}
\end{aligned} \tag{33}$$

where $\mathrm{erf}(x)$ is the error function of x. We set

$$\begin{aligned}
\rho^{(\mathrm{Gaussian})}(\beta_P) = & \frac{1}{2}[1 + \mathrm{erf}(\frac{\beta_P J}{\sqrt{2}})] - \\
& \frac{1}{2}\{1 + \mathrm{erf}[\frac{\beta_P J}{\sqrt{2}}(1-2\xi)]\}\, e^{-2(\beta_P J)^2 \xi(1-\xi)} .
\end{aligned} \tag{34}$$

The thermodynamic quantity of the node degree for bonds at node i, $[< k_{\mathrm{bond}}(i) >_T]_R^{(\mathrm{Gaussian})}$, is, using eqs. (7, 8, 10, 13, 32, 34) when $\beta = \beta_P$,

$$[< k_{\mathrm{bond}}(i) >_T]_R^{(\mathrm{Gaussian})} = \rho^{(\mathrm{Gaussian})}(\beta_P)\, k(i) . \tag{35}$$

The thermodynamic quantity of the square of the node degree for bonds at node i, $[< k_{\mathrm{bond}}^2(i) >_T]_R^{(\mathrm{Gaussian})}$, is, using eqs. (7, 8, 10, 14, 32, 34) when $\beta = \beta_P$,

$$\begin{aligned}
& [< k_{\mathrm{bond}}^2(i) >_T]_R^{(\mathrm{Gaussian})} = \\
& [\rho^{(\mathrm{Gaussian})}(\beta_P)]^2 k(i)[k(i)-1] + \\
& \rho^{(\mathrm{Gaussian})}(\beta_P)\, k(i) .
\end{aligned} \tag{36}$$

Eqs. (33, 34, 35, 36) are formulated as eqs. (19, 20, 21), respectively. Therefore, the bond does not depend on the size of $k(i)$. Using eqs. (15, 16, 17, 34, 35, 36), we obtain

$$\begin{aligned}
& (2-a)\{1 + \mathrm{erf}[\frac{\beta_P J}{\sqrt{2}}(1-2\xi)]\}\, e^{-2(\beta_P J)^2 \xi(1-\xi)} \leq \\
& (2-a)[1 + \mathrm{erf}(\frac{\beta_P J}{\sqrt{2}})] - 2a .
\end{aligned} \tag{37}$$

Ineq. (37) is given by the inequality when the cluster is percolated. Ineq. (37) is given by the equality when the cluster is at the percolation transition point. From eq. (37), the percolation threshold is shifted by adjusting ξ while eq. (32) holds. Therefore, it is realized that, by adjusting ξ, the percolation threshold agrees with the location of the multicritical point since, by adjusting ξ, the percolation threshold is shifted along the Nishimori line.

We naively estimate the location of the multicritical point for the Gaussian model on the random graphs with arbitary degree distributions. We assume that ξ is the function of a. The network is at the percolation point when a is one.[19] The exchange interaction is only the ferromagnetic interaction when a is one. Therefore, ξ is one when a is one since the algorithm is the SW algorithm[12] when ξ is one. The percolation transition point of the cluster by the SW algorithm agrees with the phase transition point in ferromagnetic spin models.[25] We set $< k >_N = N-1$, $< k^2 >_N = (N-1)^2$,

$a = 2/(N-1)$, $J_0 \to J_0/N$, and $J \to J/\sqrt{N}$. From the settings, the model on the network becomes the Sherrington-Kirkpatrick (SK) model.[22] Using eq. (37), we obtain

$$(N-2)\{1+\mathrm{erf}[\frac{\beta_P J}{\sqrt{2N}}(1-2\xi)]\}\, e^{-\frac{2(\beta_P J)^2\xi(1-\xi)}{N}} \leq$$
$$(N-2)[1+\mathrm{erf}(\frac{\beta_P J}{\sqrt{2N}})]-2\,. \quad (38)$$

From numerical estimation of eq. (38), $\xi = O(\frac{1}{\sqrt{N}})$ since β_P has a finite value only when $\xi = O(\frac{1}{\sqrt{N}})$. Therefore, we naively estimate

$$\xi^* = \sqrt{a}\,. \quad (39)$$

Using eqs. (37, 39), we obtain

$$(2-a)\{1+\mathrm{erf}[\frac{\beta_P J}{\sqrt{2}}(1-2\sqrt{a})]\} \times$$
$$e^{-2(\beta_P J)^2\sqrt{a}(1-\sqrt{a})} \leq$$
$$(2-a)[1+\mathrm{erf}(\frac{\beta_P J}{\sqrt{2}})]-2a\,. \quad (40)$$

Ineq. (40) gives the location of the multicritical point. Using eqs. (32, 38, 39), the location of the multicritical point for the SK model is numerically estimated as $J_0^* \sim 0.886 \sim \sqrt{\pi/4}$ and $T_P^* \sim 1.128 \sim \sqrt{4/\pi}$ when $J/k_B = 1$ is set. Ineq. (40) is obtained based on some assumptions. We have assumed that ξ is written as a simple form of a, and have assumed that the location of the multicritical point for the SK model has a finite value.

7. Summary

In the present article, the analitycal result of the application of Niedermayer's algorithm is obtained on the Nishimori line. It is respectively shown for the $\pm J$ model and the Gaussian model that, by adjusting the introduced parameter ξ for Niedermayer's algorithm, the percolation threshold obtained in the present article agrees with the location of the multicritical point. In the present article, we have naively estimated the locations of the multicritical points for the $\pm J$ model and the Gaussian model on the random graphs with arbitary degree distributions. The locations of the multicritical points are obtained based on some assumptions. We have assumed that ξ is written as a simple form of the aspect a of the network, and the locations of the multicritical points for the infinit models have finite values.

1) S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes: Rev. Mod. Phys. **80** (2008) 1275.
2) S. F. Edwards and P. W. Anderson: J. Phys. F **5** (1975) 965.
3) L. Viana and A. J. Bray: J. Phys. C **18** (1985) 3037.
4) M. Mézard and G. Parisi: Eur. Phys. J. B **20** (2001) 217.
5) H. Nishimori: J. Phys. C **13** (1980) 4071; Prog. Theor. Phys. **66** (1981) 1169.
6) T. Morita and T. Horiguchi: Phys. Lett. **76A** (1980) 424.
7) H. Nishimori: Prog. Theor. Phys. **76** (1986) 305; J. Phys. Soc. Jpn. **55** (1986) 3305; **61** (1992) 1011; **62** (1993) 2793.
8) T. Horiguchi and T. Morita: J. Phys. A: Math. Gen. **14** (1981) 2715.
9) T. Horiguchi: Phys. Lett. A **81** (1981) 530.
10) H. Nishimori and K. Nemoto: J. Phys. Soc. Jpn. **71** (2002) 1198.
11) P. W. Kasteleyn and C. M. Fortuin: J. Phys. Soc. Jpn. Suppl. **26** (1969) 11; C. M. Fortuin and P. W. Kasteleyn: Physica (Utrecht) **57** (1972) 536.
12) R. H. Swendsen and J. -S. Wang: Phys. Rev. Lett. **58** (1987) 86.
13) N. Kawashima and J. E. Gubernatis: Phys. Rev. E **51** (1995) 1547.
14) Y. Tomita and Y. Okabe: Phys. Rev. B **66** (2002) 180401R.
15) C. Yamaguchi and N. Kawashima: Phys. Rev. E **65** (2002) 056710.
16) C. Yamaguchi, N. Kawashima, and Y. Okabe: Phys. Rev. E **66** (2002) 036704.
17) J. Machta, C. M. Newman, and D. L. Stein: J. Stat. Phys. **130** (2008) 113.
18) L. de Arcangelis, A. Coniglio, and F. Peruggi: Europhys. Lett. **14** (1991) 515.
19) C. Yamaguchi: arXiv:0903.0438.
20) F. Niedermayer: Phys. Rev. Lett. **61** (1988) 2026.
21) S. Liang: Phys. Rev. Lett. **69** (1992) 2145.
22) D. Sherrington and S. Kirkpatrick: Phys. Rev. Lett. **35** (1975) 1792.
23) G. Toulouse: Commun. Phys. **2** (1977) 115.
24) N. Metropolis, A. Rosenbluth, M. Rosenbluth, A. Teller, and E. Teller: J. Chem. Phys. **21** (1953) 1087.
25) A. Coniglio and W. Klein: J. Phys. A: Math. Gen. **13** (1980) 2775.